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Renormalisation group calculations of the critical exponents of the n-vector model with a free surface†

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Abstract. The correlation function critical exponents $\eta_\parallel$ and $\eta_\perp$ of the semi-infinite ϕ^4 model with O(n) symmetry have been calculated to order ε^2 for the ordinary and special transitions. The results for the ordinary transition are inconsistent with the relation $\gamma_{11} = \nu - 1$ proposed by Bray and Moore. Comparisons with series expansion results in two and three dimensions are presented for $n = 0$ and $n = 1$.

1. Introduction

For spin systems of the n-vector type with a free surface and positive extrapolation length λ, two different classes of phase transition can be identified (Lubensky and Rubin 1975a, b, Bray and Moore 1977a, b). (The extrapolation length is the distance beyond the free surface where the spin profile in the ordered phase would vanish if linearly extrapolated from the boundary.) When $0 < \lambda < \infty$ the transition is referred to as the *ordinary transition* and when $\lambda^{-1} = 0$ one speaks of the *special transition*. When $\lambda < 0$, the surface orders at a higher temperature than that at which the bulk orders. Thus as the temperature is lowered, the surface orders first, corresponding to the surface transition, then, at the (lower) bulk critical temperature, the bulk orders under the influence of an ordered surface. This is called the *extraordinary transition*. We are only concerned here, however, with the case of positive extrapolation length.

Let $G(X, X')$ be the correlation function for spins at sites X and X' of a semi-infinite d-dimensional lattice. The d-dimensional vector X has components $(\boldsymbol{\rho}, z)$ where $\boldsymbol{\rho}$ is a $(d-1)$-dimensional vector parallel to the free surface at $z = 0$. Restricted translational invariance obtains, so that $G(X, X') = G(r, z, z')$, where $r = |\boldsymbol{\rho} - \boldsymbol{\rho}'|$. The critical exponents $\eta_\parallel$ and $\eta_\perp$ are defined via the asymptotic relations $G(r, z, z') \sim r^{2-d-\eta_\parallel}$ as $r \to \infty$ with z and z' fixed, and $G(r, z, z') \sim z^{2-d-\eta_\perp}$ with r and z' fixed. When $\lambda^{-1} = 0$ these exponents are normally written as $\eta_\parallel^{\text{sp}}$ and $\eta_\perp^{\text{sp}}$. The exponents γ_1 and γ_{11} (and similarly γ_1^{sp} and γ_{11}^{sp}) characterise the asymptotic behaviour—at the ordinary (or special) transition temperature—of the response of the magnetisation on the surface to a magnetic field change in the bulk, and to a magnetic field change on the surface, respectively. In terms of the correlation functions, we have

$$\sum_r G(r, 0, 0) \sim t^{-\gamma_{11}} \qquad (1.1)$$

† A preliminary account of these results appeared in Reeve and Guttmann (1980) and Reeve (1981).

and

$$\sum_{r,z'} G(r, 0, z') \sim t^{-\gamma_1} \tag{1.2}$$

while the bulk exponent γ is given by

$$\lim_{z\to\infty} \sum_{r,z'} G(r, z, z') \sim t^{-\gamma} \tag{1.3}$$

where $t = |1 - T/T_c|$ is the usual reduced temperature.

Our interest in this problem stems largely from the work of Bray and Moore (1977a, b), who proposed the relation $\gamma_{11} = \nu - 1$, where ν is the usual bulk correlation length exponent. Their proposal followed from a perturbation expansion, which was claimed to be exact to all orders, and was summed. This conjecture agrees with all known *exact* results but is in apparent conflict with certain series expansion results. The exact results include mean-field exponents, the two-dimensional Ising model, the ϕ^4, O(n) model to order ε and in the $n = \infty$ limit. Most notably, Barber *et al* (1978) found from series studies that for the self-avoiding walk model (the $n = 0$ realisation of the n-vector model) in two dimensions $\gamma_{11} = -0.19^{+0.03}_{-0.02}$, while $\nu - 1 = -0.25$, and in three dimensions $\gamma_{11} = -0.35 \pm 0.05$, which is only just consistent with the series-based estimate $\nu - 1 = -0.4$. The validity of the technique which produced the observed discrepancy in the $d = 2$ case was confirmed by Enting and Guttmann (1980), who applied an identical method of analysis, *mutatis mutandis*, to the two-dimensional Ising model, and found that the series analysis yielded a value for γ_{11} in agreement with that of $\nu - 1$.

For the three-dimensional Ising model ($n = 1$), Whittington *et al* (1980) estimated $\gamma_{11} = -0.33 \pm 0.04$, which again is just in agreement with the series estimate $\nu - 1 = -0.362^{+0.001}_{-0.002}$. For the percolation problem, which of course is not describable within the framework of ϕ^4 field theory, but for which Bray and Moore's relation should still be valid, De'Bell and Essam (1981) obtained series estimates of γ_{11} and ν which quite clearly violated the proposed relation. Their results held for both bond and site percolation on the three-dimensional FCC lattice. In two dimensions $\nu > 1$, so that the surface transition does not exist, and hence the relation is not expected to hold. In addition, they considered the two-dimensional SAW model on a different lattice from that considered by Barber *et al* and confirmed the breakdown reported earlier.

In this paper we report calculations to second order in $\varepsilon = 4 - d$ of $\eta_\parallel$, $\eta_\perp$, $\eta_\parallel^{\mathrm{sp}}$ and $\eta_\perp^{\mathrm{sp}}$. The correlation function scaling relations

$$\gamma_{11} = \nu(1 - \eta_\parallel) \tag{1.4}$$

and

$$\gamma_1 = \nu(1 - \eta_\perp) \tag{1.5}$$

of Binder and Hohenberg (1972, 1974) are then used to calculate γ_1 and γ_{11}. As shown explicitly by Diehl and Dietrich (1981), the relations (1.4) and (1.5) and the surface scaling relation

$$2\gamma_1 - \gamma_{11} = \gamma + \nu \tag{1.6}$$

(Barber 1973), as well as the analogous expressions for the special transition, are intrinsic to the renormalisation group method we have used.

Other recent work on this problem includes a thorough investigation of the scaling theory by Burkhardt and Eisenriegler (1980), and several position-space renormalisation group calculations (Ŝvrakić and Wortis 1977, Burkhardt and Eisenriegler 1977, 1978). Unfortunately, those calculations do not allow us to comment on the validity of the proposed relations $\gamma_{11} = \nu - 1$.

The organisation of the remainder of this paper is as follows: the next two sections specify the model under consideration, and outline the general renormalisation group procedure. Sections 4 and 5 give details of the calculation for the ordinary and special transitions respectively. In § 6 we briefly discuss the logarithmic corrections to the susceptibility exponents γ_1 and γ_{11} that arise in the four-dimensional semi-infinite ϕ^4 model with $O(n)$ symmetry, and take the opportunity to correct certain erroneous results contained in Guttmann and Reeve (1980). Section 7 comprises a brief summary.

2. The model

The pioneering work on the renormalisation group approach to semi-infinite ϕ^4, $O(n)$ systems was carried out by Lubensky and Rubin (1975a, b). The Hamiltonian they used is $H = H_0 + H_1$, where

$$H_0 = \tfrac{1}{2}\sum_{X}(S(X))^2 - \tfrac{1}{2}K\sum_{X,\delta}S(X)S(X+\delta) - \tfrac{1}{2}K\Delta_S\sum_{X,\delta_\parallel}S(X)S(X+\delta_\parallel), \tag{2.1}$$

$$H_1 = \tilde{g}\sum_{X}(S(X)S(X))^2. \tag{2.2}$$

The $S(X)$ are the n-dimensional spin vectors at site X of a d-dimensional hypercubic lattice, while the factor Δ_S allows for a different spin interaction strength on the surface than in the bulk (if required). The d-dimensional lattice vector $\delta = (\delta_\parallel, \delta_\perp)$, with $\delta_\parallel$ the $(d-1)$-dimensional component parallel to the surface and $\delta_\perp$ the remaining component perpendicular to the surface.

The Fourier expansion functions used to diagonalise H_0 are given by

$$\psi_q(X) = \sqrt{2}\, \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{\rho}} \sin(kz + \phi), \tag{2.3}$$

where $q = (\boldsymbol{p}, k)$, $\boldsymbol{p}$ being a $(d-1)$-dimensional vector and k a scalar, $\tan\phi = (\sin k)/(\cos k - \Delta_S \Sigma_{\delta_\parallel} \cos \boldsymbol{p}\cdot\boldsymbol{\delta}_\parallel)$, and the surface is located at $z = 0$. The total Hamiltonian transformed to the momentum representation can be written

$$H = \tfrac{1}{2}\int \mathrm{d}q\, (m^2 + q^2)\phi_i(q)\phi_i(\nu q) + \frac{1}{8}\frac{g_0}{4!}\sum_{\varepsilon_i=\pm1}\int\Big(\prod_i \mathrm{d}q_i\Big)\phi_i(q_1)\phi_i(q_2)\phi_j(q_3)\phi_j(q_4)f(q,\varepsilon) \tag{2.4}$$

where $\nu q = (-\boldsymbol{p}, k)$.

For the ordinary transition, at the point $\lambda = 1$ corresponding to $\Delta_S = 0$, we have

$$f(q,\varepsilon) = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\delta\Big(\sum_i p_i\Big)\delta\Big(\sum_i \varepsilon_i k_i\Big) \tag{2.5}$$

while for the special transition

$$f(q,\varepsilon) = \delta\Big(\sum_i p_i\Big)\delta\Big(\sum_i \varepsilon_i k_i\Big)\operatorname{sgn}(k_1k_2k_3k_4) \tag{2.6}$$

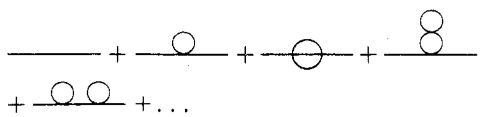
and in that case (2.3) reduces to

$$\psi_q(X) = \sqrt{2}\, e^{i\boldsymbol{p}\cdot\boldsymbol{\rho}} \cos(kz)\, \mathrm{sgn}(k) \tag{2.7}$$

since in this case $\tan\phi \sim \lambda \sin k$, hence $\phi = \frac{1}{2}\pi\, \mathrm{sgn}(k)$.

3. Renormalisation group procedure

The renormalisation method we have used is the principle of minimal subtraction of the divergent cut-off dependent terms from the Green functions, as outlined by Amit (1978) for the isotropic, infinite case. We begin by calculating the two-point momentum-space Green functions $G_p(k_1, k_2)$ to second order in the coupling constant and evaluating diagrams within the framework of dimensional regularisation. The finite quantity $G_p^{\mathrm{R}}(k_1, k_2)$ is found by minimal subtraction of poles in ε from $G_p(k_1, k_2)$. The two quantities are related by $G_p(k_1, k_2) = ZG_p^{\mathrm{R}}(k_1, k_2)$ where $Z^{1/2}$ is the wavefunction renormalisation factor. Since all the poles in ε contained in Z originate only from the momentum conserving or 'bulk' term in $G_p(k_1, k_2)$, the diagonal part of $G_p(k_1, k_2)$, which is proportional to $[\delta(q_1 - \nu q_2) - \delta(q_1 + q_2)]$, obeys the usual renormalisation group equations. Consequently, and because the mass and coupling constant renormalisation functions remain exactly as for the bulk system (as becomes clear when they are explicitly calculated, as discussed above (4.5)), the bulk exponents can be calculated in the usual way (Amit 1978). The renormalised coupling constant and its fixed point value can therefore be taken from the calculation for the infinite system, and we have gleaned these quantities directly from Amit (1978) after suitably matching conventions. The next step is to calculate the inverse Fourier transform of $G_p^{\mathrm{R}}(k_1, k_2)$, to give $G_p^{\mathrm{R}}(z_1, z_2)$ in the mixed space, and subtracting off any divergent cut-off dependence which occurs in the inversion to make $G_p^{\mathrm{R}}(z_1, z_2)$ finite. The procedure is rigorously justified in Diehl and Dietrich (1980, 1981). The decay of the correlations of spins in the boundary surface is assumed to behave like $G_p^{\mathrm{R}}(0, 0) \sim p^{-1+\eta_\parallel}$, while the asymptotic form of the bulk-spin–surface-spin correlations is assumed to be $G_p^{\mathrm{R}}(0, z_2 \to \infty) \sim p^{-2+\eta_\perp}$ as $p \to 0$. This last relation is of lesser utility than calculating $\Sigma_{z_2 \geq 0}\, G_p^{\mathrm{R}}(0, z_2)$, as the sum can be performed prior to taking the Fourier transform and has the same asymptotic form as $G_p^{\mathrm{R}}(0, z_2 \to \infty)$. The critical exponents are then identified by exponentiation. Diagrammatically the bare Green function is

$G_p(k_1, k_2) =$ 

which is explicitly

$$G_p(k_1, k_2) = G_p^0(k_1, k_2) - \left(\frac{n+2}{6}\right) g_0 A + \frac{n+2}{18} g_0^2 B + \left(\frac{n+2}{6}\right)^2 g_0^2 C + \left(\frac{n+2}{6}\right)^2 g_0^2 D + \dots \tag{3.1}$$

where $G_p^0(k_1, k_2)$ is the bare propagator and the capitals represent the values of the integrals corresponding to each diagram above.

4. The ordinary transition

The propagator for this system is $G_p^0(k_1, k_2) = [\delta(q_1 - \nu q_2) - \delta(q_1 + q_2)]/2q_1^2$ and the contribution of each diagram up to two loops is given below.

$$q_1^2 q_2^2 A = -\delta' \frac{g'}{4}\left(1+\frac{\varepsilon}{2}\right)(1-\varepsilon \ln 2)(|\sigma_+| - |\sigma_-| - \varepsilon|\sigma^+| \ln|\sigma^+| + \varepsilon|\sigma^-| \ln|\sigma^-|), \tag{4.1}$$

$$\begin{aligned} q_1^2 q_2^2 B = & -\delta \frac{g'^2}{8}\left(\frac{9}{2} q_1^2 + q_1^2 \ln q_1^2 - \frac{q_1^2}{\varepsilon}\right) - \frac{3\delta' g'^2}{4\varepsilon}(|\sigma^+| - |\sigma^-|) \\ & + \tfrac{3}{4}\delta' g'^2(|\sigma^+| \ln|\sigma^+| - |\sigma^-| \ln|\sigma^-|) - \tfrac{3}{4}\delta' g'^2(1-\ln 2)(|\sigma^+| - |\sigma^-|) \\ & + \tfrac{3}{8}\delta' g'^2(|\sigma_+| - |\sigma_-|) \ln[p_1^2 + (|\sigma_+| + |\sigma_-|)^2] \\ & + \tfrac{3}{4}\delta' g'^2 \frac{\sigma_+^2 - \sigma_-^2}{p_1} \tan^{-1}\left(\frac{p_1}{|\sigma_+| + |\sigma_-|}\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} q_1^2 q_2^2 C = & -\frac{\delta' g'^2}{4\varepsilon}(|\sigma^+| - |\sigma^-|) - \frac{\delta' g'^2}{8}(|\sigma^+| - |\sigma^-|)(3 - 4\ln 2) \\ & + \tfrac{1}{2} g'^2 \delta'(|\sigma^+| \ln|\sigma^+| - |\sigma^-| \ln|\sigma^-|), \end{aligned} \tag{4.3}$$

$$\begin{aligned} q_1^2 q_2^2 D = & \frac{\delta' g'^2}{16}\Bigg[2(|\sigma_+| - |\sigma_-|) - (|\sigma_+| - |\sigma_-|) \ln[(|\sigma_+| - |\sigma_-|)^2 + p_1^2] \\ & - \frac{2(\sigma_+^2 - \sigma_-^2)}{p_1} \tan^{-1}\left(\frac{p_1}{|\sigma_+| + |\sigma_-|}\right) - 2p_1 \tan^{-1}\left(\frac{|\sigma_+| - |\sigma_-|}{p_1}\right)\Bigg], \end{aligned} \tag{4.4}$$

where $\delta = [\delta(q_1 - \nu q_2) - \delta(q_1 + q_2)]/2$, $\delta' = \delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)$, $g' = 2\pi^{2-\varepsilon/2} g_0/\Gamma(2 - \varepsilon/2)$ and $\sigma_\pm = (k_1 \pm k_2)/2$.

Putting these values into the sum (3.1) and subtracting the poles in ε with residue proportional to δ arising in B gives a finite quantity $G_p^R(k_1, k_2)$. In table 1 we list all the Fourier transforms needed to extract the low-momentum behaviour from $G_p^R(k_1, k_2)$ for both $G_p^R(0, 0)$ and $\Sigma_{z_2} G_p^R(0, z_2)$. The method of obtaining these transforms is outlined by example in the Appendix.

Table 1. Fourier transforms needed to recover the low-momentum behaviour of $G_p^R(0, 0)$ and $\Sigma_{z_2} G_p^R(0, z_2)$ from $G_p^R(k_1, k_2)$ for the ordinary transition.

Function $\times q_1^2 q_2^2/\delta'$	Contribution to $G_p^R(0, 0)2\pi/p$	Contribution to $\Sigma_{z_2} G_p^R(0, z_2)2\pi p$
$\delta'\delta$	-1	1
$\|\sigma_+\| - \|\sigma_-\|$	$-4 + 4\ln 2 + 4\ln p$	$2 - 4\ln 2 - 2\ln p$
$\|\sigma_+\| \ln\|\sigma_+\| - \|\sigma_-\| \ln\|\sigma_-\|$	0	$-2(\ln 2)\ln p$
$q_1^2 \ln q_1^2 \delta\delta'$	$-2\ln p$	$2\ln p$
$(\|\sigma_+\| - \|\sigma_-\|) \ln[p^2 + (\|\sigma_+\| + \|\sigma_-\|)^2]$	$8(\ln p)(\ln 2 - 1)$	$4(\ln p)(1 - 2\ln 2)$
$(\|\sigma_+\| - \|\sigma_-\|) \ln[p^2 + (\|\sigma_+\| - \|\sigma_-\|)^2]$	$16(\ln p)(\ln 2 - 1)$	$4(\ln p)(1 - 3\ln 2)$
$p \tan^{-1} \frac{\|\sigma_+\| - \|\sigma_-\|}{p}$	$4(\ln 2)\ln p$	$-2(\ln 2)\ln p$
$\frac{\sigma_+^2 - \sigma_-^2}{p} \tan^{-1}\left(\frac{p}{\|\sigma_+\| + \|\sigma_-\|}\right)$	$4\ln p$	$-2\ln p$

Exponentiating $G_p^{\mathrm{R}}(0,0)$ and $\Sigma_{z_2} G_p^{\mathrm{R}}(0, z_2)$ then gives

$$\eta_\parallel = 2 - \frac{n+2}{n+8}\varepsilon - \frac{(n+2)(17n+76)}{2(n+8)^3}\varepsilon^2 \tag{4.5}$$

and

$$\eta_\perp = 1 - \frac{n+2}{2(n+8)}\varepsilon - \frac{(n+2)(4n+17)}{(n+8)^3}\varepsilon^2 \tag{4.6}$$

which together with (1.4) and (1.5) give

$$\gamma_1 = \frac{1}{2} + \frac{n+2}{2(n+8)}\varepsilon + \frac{(n+2)(2n^2+49n+144)}{8(n+8)^3}\varepsilon^2 \tag{4.7}$$

and

$$\gamma_{11} = -\frac{1}{2} + \frac{n+2}{4(n+8)}\varepsilon + \frac{(n+2)(n^2+31n+124)}{8(n+8)^3}\varepsilon^2. \tag{4.8}$$

The bulk ν value (for instance see Amit (1978)) is

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\varepsilon + \frac{(n+2)(n^2+23n+60)}{8(n+8)^3}\varepsilon^2. \tag{4.9}$$

Inspection shows that the relation $\gamma_{11} = \nu - 1$ is violated at second order in ε. The results (4.5) and (4.6) have been independently obtained by Diehl and Dietrich (1980). A further check is given by the consistency of our results with equation (1.6).

In order to see the effect of these new terms in the ε expansion for γ_1 and γ_{11}, we show in table 2 the sums to order ε and ε^2 of γ_1 and γ_{11}, as well as the best series estimates when $n = 0$ and $n = 1$. In every case the $O(\varepsilon^2)$ term has effected a substantial improvement over the sum to $O(\varepsilon)$, and in three dimensions all sums to $O(\varepsilon^2)$ are within 3% of series calculations.

Table 2. Sum to order ε and ε^2 of γ_1 and γ_{11}, compared with the best series estimates for $n = 0$ and $n = 1$ in two and three dimensions. Values marked with an asterisk are exact. [a] Barber *et al* (1978), [b] Enting and Guttmann (1980).

	n	d	Sum to $O(\varepsilon)$	Sum to $O(\varepsilon^2)$	Best series
γ_1	0	2	0.75	1.031	0.945[a]
	1	2	0.833	1.235	1.375*
	0	3	0.625	0.695	0.70[a]
	1	3	0.667	0.767	0.78[b]
γ_{11}	0	2	−0.375	−0.133	−0.19[a]
	1	2	−0.333	−0.012	0*
	0	3	−0.438	−0.377	−0.35[a]
	1	3	−0.417	−0.336	−0.33[b]

5. The special transition

In the special transition the system described by equations (2.4), (2.6) and (2.7) has propagator $G_p^0(k_1, k_2) = \mathrm{sgn}(k_1 k_2)[\delta(q_1 - \nu q_2) + \delta(q_1 + q_2)]/2q_1^2$. The renormalisation procedure is as described in § 3, and the relevant integrals in (3.1) become

$$q_1^2 q_2^2 A = -\tfrac{1}{4} g' \delta''(1+\tfrac{1}{2}\varepsilon)(1-\varepsilon \ln 2)(|\sigma^+| + |\sigma^-| - \varepsilon|\sigma^+| \ln|\sigma^+| - \varepsilon|\sigma_-| \ln|\sigma_-|), \tag{5.1}$$

$$q_1^2 q_2^2 B = \tfrac{1}{16}\delta''' g'^2(-q_1^2/\varepsilon + q_1^2 \ln q_1^2 - \tfrac{9}{2} q_1^2) + \tfrac{3}{4}\delta'' g'^2 \Big[-(1/\varepsilon)(|\sigma_+| + |\sigma_-|)$$
$$- (2-\ln 2)(|\sigma_+| + |\sigma_-|) + |\sigma_+| \ln|\sigma_+| + |\sigma_-| \ln|\sigma_-|$$
$$+ \tfrac{1}{2}(|\sigma_+| + |\sigma_-|) \ln[p_1^2 + (|\sigma_+| + |\sigma_-|)^2] + p_1 \tan^{-1}\left(\frac{p_1}{|\sigma_+| + |\sigma_-|}\right)\Big], \tag{5.2}$$

$$q_1^2 q_2^2 C = -\tfrac{1}{4}\delta'' g'^2 [(1/\varepsilon)(|\sigma_+| + |\sigma_-|) + (\tfrac{1}{2} - 2\ln 2)(|\sigma_+| + |\sigma_-|)$$
$$+ 2(|\sigma_+| \ln|\sigma_+| + |\sigma_-| \ln|\sigma_-|)], \tag{5.3}$$

$$q_1^2 q_2^2 D = \delta'' g'^2 \Big(-\tfrac{1}{8}(|\sigma_+| + |\sigma_-|) - \tfrac{1}{8} p_1 \tan^{-1}\left(\frac{p_1}{|\sigma_+| + |\sigma_-|}\right)$$
$$+ \tfrac{1}{16}(|\sigma_+| + |\sigma_-|) \ln[p_1^2 + (|\sigma_+| + |\sigma_-|)^2]$$
$$- \tfrac{1}{16}(|\sigma_+| + |\sigma_-|) \ln[p_1^2 + (|\sigma_+| - |\sigma_-|)^2] - \frac{1}{8}\frac{\sigma_+^2 - \sigma_-^2}{p_1} \tan^{-1}\left(\frac{p_1}{|\sigma_+| - |\sigma_-|}\right)\Big), \tag{5.4}$$

where $\delta''' = \mathrm{sgn}(k_1 k_2)[\delta(q_1 - \nu q_2) + \delta(q_1 + q_2)]/2$ and $\delta'' = \mathrm{sgn}(k_1 k_2)\delta(\boldsymbol{p}_1 + \boldsymbol{p}_2)$.

The Fourier transforms needed to find the low-momentum behaviour of $G_p^R(0, 0)$ and $\Sigma_{z_2} G_p^R(0, z_2)$ are given in table 3. Exponentiation of the Green functions allows the identification

$$\eta_\parallel = -\frac{n+2}{n+8}\varepsilon - \frac{5(n+2)(n-4)}{2(n+8)^3}\varepsilon^2 \tag{5.5}$$

and

$$\eta_\perp = -\frac{n+2}{n+8}\varepsilon - \frac{(n+2)(n-7)}{(n+8)^3}\varepsilon^2. \tag{5.6}$$

Table 3. Fourier transforms needed to recover the low-momentum behaviour of $G_p^R(0, 0)$ and $\Sigma_{z_2} G_p^R(0, z_2)$ for the special transition.

Function $\times q_1^2 q_2^2/\delta''$	Contribution to $G_p^R(0, 0) 2\pi p$	Contribution to $\Sigma_{z_2} G_p^R(0, z_2) 2\pi p^2$
$\delta''\delta'$	1	1
$\|\sigma_+\| + \|\sigma_-\|$	$-4 \ln 2 - 4 \ln p$	$-2 \ln p$
$\|\sigma_+\| \ln\|\sigma_+\| + \|\sigma_-\| \ln\|\sigma_-\|$	0	$2(\ln 2) \ln p$
$p \tan^{-1} \dfrac{p}{\|\sigma_+\| + \|\sigma_-\|}$	0	0
$(\|\sigma_+\| + \|\sigma_+\|) \ln[p^2 + (\|\sigma_+\| + \|\sigma_-\|)^2]$	$-8(\ln 2) \ln p$	0
$q_1^2 \ln q_1^2 \delta''\delta'$	$2 \ln p$	$2 \ln p$
$(\|\sigma_+\| + \|\sigma_-\|) \ln[p^2 + (\|\sigma_+\| - \|\sigma_-\|)^2]$	$-16(\ln 2) \ln p$	0
$\dfrac{\sigma_+^2 + \sigma_-^2}{p} \tan^{-1}\left(\dfrac{p}{\|\sigma_+\| - \|\sigma_-\|}\right)$	$-4(\ln 2) \ln p$	0

These results have been independently verified (Diehl and Dietrich, private communication) and are consistent with the surface scaling relation (1.6).

Unfortunately, series expansion results at the special transition are marred by cross-over effects (Binder and Hohenberg 1974) and so a direct comparison is inappropriate.

6. Exponents in four dimensions

In an earlier paper (Guttmann and Reeve 1980), we obtained expressions for the susceptibilities χ_1 and χ_{11} for the case of space dimensionality equal to four. As pointed out by Diehl and Dietrich (1981), our results are incorrect because additional singularities in the renormalised Green function arise when z/ξ—the distance from the surface scaled by the correlation length—vanishes.

With this correction, our results become

$$\chi_1(t) \sim At^{-1/2}|\ln t|^{(n+2)/(n+8)} \tag{6.1}$$

and

$$\chi_{11}(t) \sim Bt^{1/2}|\ln t|^{(n+2)/2(n+8)}. \tag{6.2}$$

This result can be confirmed by an alternative derivation parallelling that given by Essam *et al* (1978) for percolation at the critical dimension. From the results of Rudnick and Nelson (1976), it follows that near $d=4$ we can write

$$\chi_1 = A\left[1+\left(1-\frac{B}{\varepsilon\Delta_1}\right)(t^{\varepsilon\Delta_1}-1)\right]^{\theta_{\gamma_1}} t^{-\gamma_1(\varepsilon)} \tag{6.3}$$

and

$$\chi_{11} = B\left[1+\left(1-\frac{C}{\varepsilon\Delta_1}\right)(t^{\varepsilon\Delta_1}-1)\right]^{\theta_{\gamma_{11}}} t^{-\gamma_{11}(\varepsilon)} \tag{6.4}$$

where $\gamma_1(\varepsilon)$ and $\gamma_{11}(\varepsilon)$ are given by (4.7) and (4.8) respectively. The exponent Δ_1 is the usual correction-to-scaling exponent defined through

$$\chi_1 \sim t^{-\gamma_1(\varepsilon)} + at^{-\gamma_1(\varepsilon)+\varepsilon\Delta_1} \tag{6.5}$$

and

$$\chi_{11} \sim t^{-\gamma_1(\varepsilon)} + bt^{-\gamma_{11}(\varepsilon)+\varepsilon\Delta_1}. \tag{6.6}$$

Comparing (6.5) and (6.6) with (8.6) of Brézin *et al* (1976) yields $\Delta_1=\frac{1}{2}$. Now since $\chi_1 \sim t^{-1/2}$ and $\chi_{11} \sim t^{1/2}$ for $\varepsilon<0$, (6.3) and (6.4) yield $\theta_{\gamma_1}=\gamma_1'(0)/\Delta_1=2\gamma_1'(0)$ and $\theta_{\gamma_{11}}=2\gamma_{11}'(0)$. Letting $\varepsilon\to 0^+$ in (6.3) and (6.4) then gives $\chi_1 \sim t^{-1/2}|\ln t|^{\theta_{\gamma_1}}$ and $\chi_{11} \sim t^{1/2}|\ln t|^{\theta_{\gamma_{11}}}$. From (4.7) and (4.8) we therefore obtain the results for θ_{γ_1} and $\theta_{\gamma_{11}}$ given in (6.1) and (6.2).

Given this correction, the amplitudes quoted in our earlier paper also require modification. The corrected results are, for the case $n=0$,

$$kT\chi_1(v)/m^2 \sim 2.2(1-v/v_c)^{-1/2}|\ln(1-v/v_c)|^{1/4}$$

and

$$kT\chi_{11}(v)/m^2 \sim 6.1-12.8(1-v/v_c)^{1/2}|\ln(1-v/v_c)|^{1/8}.$$

7. Summary

Using renormalised perturbation theory, expansions in powers of ε to order ε^2 have been presented for the correlation function exponents $\eta_\parallel$ and $\eta_\perp$ of a semi-infinite ϕ^4, O(n) system. Both the ordinary and special transitions have been considered. For the ordinary transition the results do not agree with the relation $\gamma_{11} = \nu - 1$ of Bray and Moore, while comparisons with series expansion results show good agreement. Recently Burkhardt and Eisenriegler (1980) have studied the phase diagram and renormalisation group flow properties for a more general model that includes the system studied here, and have elucidated the exponent relations which must hold if the relation $\gamma_{11} = \nu - 1$ is to be correct—notably that the singularity α_1^{SB} characterising the dominant behaviour of the thermal singularity of the semi-infinite free energy density, $t^{1-\alpha_1^{SB}}$, must be equal to the bulk specific heat exponent α. Our results, for both the ordinary and special transition, have been confirmed by Diehl and Dietrich (1980, 1981 and private communication).

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Appendix. A calculation of one of the Fourier transforms of table 1

The inverse Fourier transform of the function $\delta(\boldsymbol{p}_1+\boldsymbol{p}_2)(|\sigma_+|-|\sigma_-|)/q_1^2q_2^2$ for the ordinary case when $z_1 = z_2 = 0$ is, after manipulating the region of integration,

$$F = \frac{4}{\pi^2}\int_0^\Lambda \mathrm{d}k_1 \int_0^{k_1} \mathrm{d}k_2\, k_2 \frac{\sin k_1 \sin k_2}{(p_1^2+k_1^2)(p_1^2+k_2^2)}. \qquad \text{(A1)}$$

We can in this case (and some others) perform this integral with $\Lambda = \infty$. However, this is not always the case, and so we elect to keep Λ finite and take only the leading terms in the $\sin k_1$ functions to extract the low-p_1 behaviour. Any Λ-dependent parts are subsequently discarded, as the result must be independent of Λ. So we require

$$\begin{aligned} F &= \frac{4}{\pi^2}\int_0^\Lambda \frac{\mathrm{d}k_1\, k_1}{k_1^2+p_1^2}\int_0^{k_1} \frac{\mathrm{d}k_2\, k_2}{(p_1^2+k_1^2)} \\ &= \frac{4}{\pi^2}\int_0^\Lambda \frac{\mathrm{d}k_1\, k_1}{k_1^2+p_1^2}[\ln(p_1^2+k_1^2)-\ln p_1^2] \\ &\sim (2/\pi)p_1(-1+\ln 2+\ln p_1) \qquad \text{for small } p_1. \end{aligned}$$

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